A sign error in the Minkowski space-time plot and its consequences

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Abstract

A sign error in an angle while drawing the original Minkowski plot has persisted for a century in text books and the pedagogical literature. When it is corrected, the 'length contraction' effect derived from the geometry of the plot disappears. It is also shown how the 'relativity of simultaneity' effect that has been derived from the plot results from a lack of correspondence between certain geometrical projections on the plot and the properties of the physical system —two spatially separated and synchronised clocks in a common inertial frame— that they are purported to describe.

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1 Introduction

Errors, originating in Einstein's 1905 special relativity paper [1], in the standard text book interpretation of the physics of the space-time Lorentz transformation (LT), have been pointed out in a series of recent papers by the present author [2, 3, 4]. In these papers it is shown that the 'length contraction' and 'relativity of simultaneity' effects, derived directly from the LT, are spurious, resulting from a failure to include important additive constants in the equations. Einstein pointed out the necessity to include such constants, to correctly describe synchronised clocks, in Ref. [1], but never actually did so himself. Since the demonstration of the spurious nature of the effects is simple, straightforward and brief, it is recalled, for the reader's convenience, in the following section of the present paper.

The remainder of this paper is devoted to a discussion of the physics of the Minkowski space-time plot. In his original 'space-time' paper [5] Minkowski derived a 'length contraction' effect from the geometry of the plot without considering directly the LT. Similar derivations are to be found in many text books on special relativity or classical electromagnetism. In Section 3, the projective-geometrical properties of the space-time plot are derived from the LT. In Section 4, Minkowski's original derivation of 'length contraction' is reviewed, and shown to result from an erroneous assumption concerning the direction of the world line of the considered, uniformly moving, object. In fact, the world line corresponding to Minkowski's choice of x' and t' axes is $x = -vt$, whereas it is assumed to be $x = vt$ and therefore to lie along the t' axis. The same mistake is made in all (with, to the present writer's best knowledge, a single exception) text-book treatments of the problem. Some examples are discussed in Section 5. Also discussed in Section 5 is the fortuitously correct derivation of time dilatation from the standard, incorrect, Minkowski plot, as well as the illusory nature of the 'relativity of simultaneity' effect suggested by superficial inspection of the plot. This error is not, as in the case of 'length contraction', the result of a trivial geometrical mistake, but arises from a naive interpretation of purely mathematical projection operations on the plot that are unrelated to the basic physics of the problem —observation in different inertial frames of events of two spatially-separated and synchronised clocks. Section 6 contains a brief summary.

2 The physics of the space-time Lorentz transformation: time dilatation and invariant lengths

The LT relates space and time coordinates as measured in two inertial frames in relative motion. In the following, it is assumed that the frame S' moves with velocity v relative to S, along the direction of a common x-x'axis. In particular, the world lines of the origin, O', of S' with spatial coordinates x',y',z'=0,0,0 and the fixed point, P', with x',y',z' = L',0,0 are considered. The LT relating the space-time coordinates of O' in S and S' is:

$$x'(\mathrm{O}') = \gamma[x(\mathrm{O}') - vt] = 0 \tag{2.1}$$

$$t'(\mathrm{O}') = \gamma[t - \frac{vx(\mathrm{O}')}{c^2}] \tag{2.2}$$

where $\beta \equiv v/c$, $\gamma \equiv 1/\sqrt{1-\beta^2}$. The space transformation equation (2.1) is equivalent to:

$$x'(\mathrm{O}') = 0 \tag{2.3}$$

giving the fixed position of O' in S', and

$$x(\mathrm{O}') = vt \tag{2.4}$$

which is the equation of motion (or world line) of O' in S. The LT relating the space and time coordinates of P' in S and S' is:

$$x'(\mathrm{P}') - \mathrm{L}' = \gamma[x(\mathrm{P}') - \mathrm{L} - vt] = 0 \tag{2.5}$$

$$t'(\mathrm{P}') = \gamma[t - \frac{v(x(\mathrm{P}') - \mathrm{L})}{c^2}] \tag{2.6}$$

where (2.5) is equivalent to:

$$x'(\mathrm{P}') = \mathrm{L}' \tag{2.7}$$

and

$$x(\mathrm{P}') = vt + \mathrm{L} \tag{2.8}$$

giving, respectively, the position of P' in S' and its equation of motion in S. In these equations t is the time recorded by a clock at rest at an arbitary position in S [1] while $t'(\mathrm{O}')$ and $t'(\mathrm{P}')$ are the times recorded by similar clocks at O'and L' in S', as observed in S. The clocks are set so that when $t = 0$, then $x(\mathrm{O}') = 0$, $x(\mathrm{P}') = \mathrm{L}$ and $t'(\mathrm{O}') = t'(\mathrm{P}') = 0$, so the clocks in S' are synchronised at this instant. (2.1) and (2.2) are recovered from (2.5) and (2.6) when $\mathrm{L} = \mathrm{L}' = 0$. It follows from (2.8) that:

$$\mathrm{L} = x(\mathrm{P}')|_{t=0} \tag{2.9}$$

L is therefore a constant that is independent of v, defined only by the choice of coordinate origin in S. As $v \to 0$, $\gamma \to 1$, S $\to$ S' and $x \to x'$, so that for $v = 0$, (2.5) is written:

$$x'(\mathrm{P}') - \mathrm{L}' = x'(\mathrm{P}') - \mathrm{L} \tag{2.10}$$

so that

$$\mathrm{L}' = \mathrm{L} \tag{2.11}$$

It then follows from (2.3),(2.4),(2.7) and (2.8) that:

$$x'(\mathrm{P}') - x'(\mathrm{O}') = x(\mathrm{P}') - x(\mathrm{O}') = \mathrm{L} \tag{2.12}$$

The spatial separation of P' and O' is therefore the same in S' and S for all values of $x(\mathrm{P}')$ and t —there is no 'relativistic length contraction' effect.

Using (2.4) to eliminate $x(\mathrm{O}')$ from (2.2), and (2.8) to eliminate $x(\mathrm{P}')$ from (2.6) gives the Time Dilatation (TD) relations:

$$t = \gamma t'(\mathrm{O}') = \gamma t'(\mathrm{P}') \tag{2.13}$$

The clocks at O' and P' therefore remain synchronised at all times: $t'(\mathrm{O}') = t'(\mathrm{P}')$ —there is no 'relativity of simultaneity' effect.

[1] As usual, an array of synchronised clocks in S may be introduced so that the comparison of t with $t'(\mathrm{O}')$ and $t'(\mathrm{P}')$ can be performed locally.

The spurious 'length contraction' effect as derived from the LT in Einstein's 1905 special relativity paper [1], and the associated 'relativity of simultaneity' effect are the consequence of using, in the present problem, an incorrect LT to describe the world line of P'. Instead of (2.5) and (2.6), the LT (2.1) and (2.2), appropriate for O', is used also for P', thus neglecting the additive constants that must be, according to Einstein [6], added to the right sides of the latter equations in order to correctly describe a synchronised clock at $x' = \mathrm{L}$. This gives:

$$x'(\mathrm{P}') = \gamma[x(\mathrm{P}') - vt] = \mathrm{L}' \tag{2.14}$$

$$t'(\mathrm{P}') = \gamma[t - \frac{vx(\mathrm{P}')}{c^2}] \tag{2.15}$$

Combining (2.14) and (2.1) gives, at any instant in S:

$$x'(\mathrm{P}') - x'(\mathrm{O}') = \mathrm{L}' = \gamma[x(\mathrm{P}') - x(\mathrm{O}')] = \gamma\mathrm{L} \tag{2.16}$$

This is the spurious 'length contraction' effect. A universal sign error in drawing the axes in the Minkowski plot has resulted, as shown below, in the same false prediction. Combining (2.15) with (2.2), in which $t(\mathrm{O}') = t(\mathrm{P}') = t$, gives

$$t'(\mathrm{P}') - t'(\mathrm{O}') = -\frac{\gamma vx(\mathrm{P}')}{c^2} \tag{2.17}$$

Events which are simultaneous in S are then not so in S'. This is the 'relativity of simultaneity' effect. How the same spurious effect results from misinterpretation of the Minkowski space-time plot is explained below.

3 Projective geometry of the space-time Lorentz transformation

It is assumed that the LT is used to describe space and time measurements of two objects, O1 and O2, that are at rest along the x' axis of the frame S', separated by a distance L$'$, O1 being at the origin of S' and O2 at $x' = \mathrm{L}'$. As above, the frame S' moves with uniform velocity v along the positive x-axis of the frame S, the x and x' axes being parallel. Space and time measurements of O1 in the frames S and S' are related by the LT:

$$x' = \gamma(x - \beta x_0) = 0 \tag{3.1}$$

$$x'_0 = \gamma(x_0 - \beta x) \tag{3.2}$$

where $x_0 \equiv ct$ and $x'_0 \equiv ct'$. These equations may be written as two-dimensional rotations by introducing the variables:

$$\cos\theta \equiv \frac{1}{\sqrt{1+\beta^2}} \tag{3.3}$$

$$\sin\theta \equiv \frac{\beta}{\sqrt{1+\beta^2}} \tag{3.4}$$

$$X' \equiv x'\sqrt{\frac{1-\beta^2}{1+\beta^2}} \tag{3.5}$$

$$X'_0 \equiv x'_0\sqrt{\frac{1-\beta^2}{1+\beta^2}} \tag{3.6}$$

as

$$X' = x\cos\theta - x_0\sin\theta \tag{3.7}$$
$$X'_0 = x_0\cos\theta - x\sin\theta \tag{3.8}$$

These equations show that the LT of both space and time coordinates may be written as the product of a two-dimensional rotation and a scale transformation, the scale factor being $\sqrt{1+\beta^2}/\sqrt{1-\beta^2}$ for both coordinates. It is easily shown that the X'-axis is obtained from the x-axis by clockwise rotation through the angle $\theta = \arctan\beta$, while the X'_0-axis is obtained from the x_0-axis by anti-clockwise rotation through the same angle. Fig.1 shows the $x, x_0 \rightarrow X'_0$ transformation of Eqn(3.8).

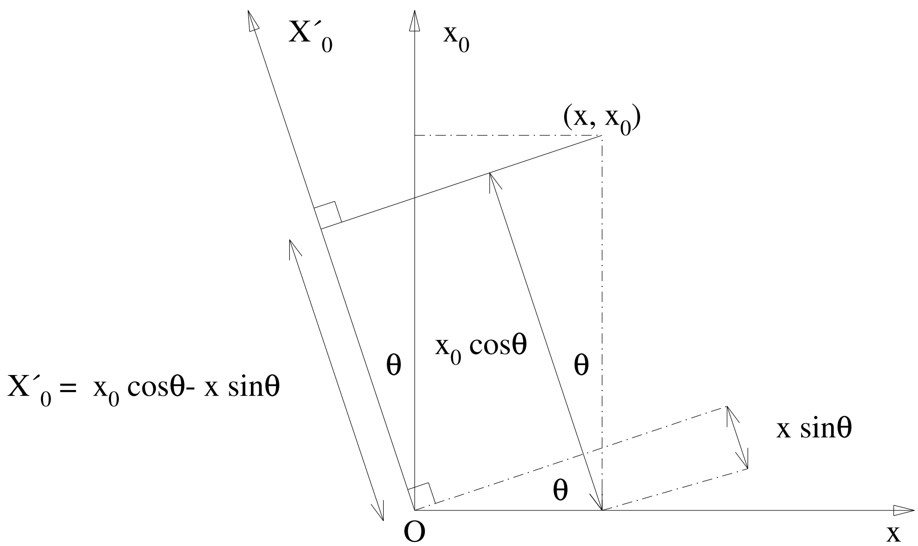

Figure 1: *The Lorentz transformation from x,x_0 to X'_0 using Eqn(3.8).*

In Fig.2 is shown the world line WL1 of O1: $\Delta x = \beta\Delta x_0$ on a Minkowski plot where x and x_0 are described by rectangular Cartesian coordinates. The corresponding X' , X'_0 axes are also drawn. It can be seen that the projection needed to obtain $\Delta x'_0$ from Δx and Δx_0 is neither orthogonal (perpendicular to the X'_0 axis) nor oblique (parallel to the X' axis). The geometry of Fig.2 and the relation (3.6) gives, for the projection angle, ϕ_t:

$$\tan\phi_t = \frac{2\beta}{\sqrt{1-\beta^2}(\sqrt{1+\beta^2}-\sqrt{1-\beta^2})} \tag{3.9}$$

This angle takes the limiting value $\pi/2$ in both the $\beta \rightarrow 0$ and $\beta \rightarrow 1$ limits. Differentiation of (3.9) shows that ϕ_t takes its minimum value when $\beta = \beta_{min}$ where:

$$(\beta_{min})^6 + (\beta_{min})^4 + (\beta_{min})^2 - 1 = 0 \tag{3.10}$$

Solving this cubic equation for $(\beta_{min})^2$ gives $\beta_{min} = 0.737$ corresponding to $\phi_t^{min} = 75.44°$.

From the geometry of Fig.2 and Eqn(3.6),

$$\Delta x'_0 = \sqrt{\frac{1+\beta^2}{1-\beta^2}}\Delta X'_0 = \sqrt{\frac{1+\beta^2}{1-\beta^2}}\left[\sqrt{(\Delta x)^2+(\Delta x_0)^2}\right]\cos 2\theta$$

$$= \frac{1+\beta^2}{\sqrt{1-\beta^2}}\Delta x_0(\cos^2\theta - \sin^2\theta) = \frac{\Delta x_0(1-\beta^2)}{\sqrt{1-\beta^2}}$$
$$= \frac{\Delta x_0}{\gamma} \quad (3.11)$$

so that $\Delta x_0 = \gamma \Delta x'_0$, the well-known and experimentally-verified time dilatation (TD) effect.

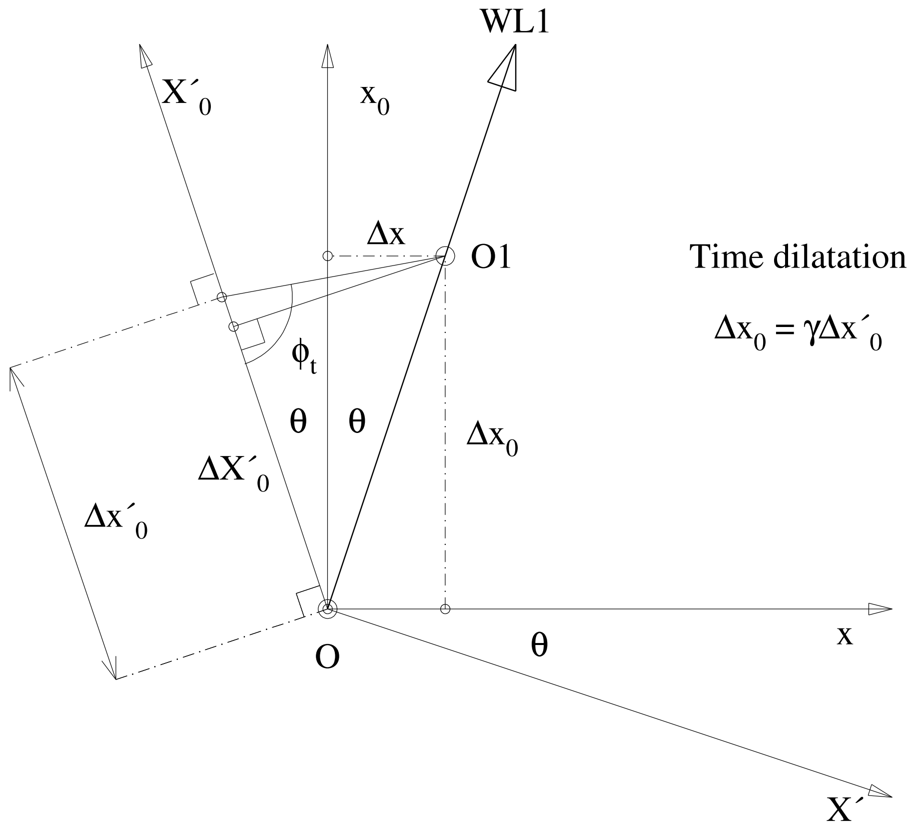

Figure 2: *Transformation of the intervals Δx and Δx_0 on the world line $\Delta x = \beta \Delta x_0 = \Delta x_0 \tan\theta$ in the frame S into the frame S' according to Eqns(3.6)-(3.8). In this and subsequent figures large circles denote world-line points and small circles projected coordinates. The LT is equivalent to an orthogonal projection onto the X'_0 axis multiplied by the scale factor of Eqn(3.6). This gives the time dilatation relation (3.11).*

The space-time LTs for O1 and O2 may be written (c.f. Eqns(2.1),(2.2),(2.5),(2.6) and (2.11)) as:

$$x'(\mathrm{O1}) = \gamma[x(\mathrm{O1}) - \beta x_0] = 0 \quad (3.12)$$
$$x'_0 = \gamma[x_0 - \beta x(\mathrm{O1})] \quad (3.13)$$
$$x'(\mathrm{O2}) - \mathrm{L} = \gamma[x(\mathrm{O2}) - \mathrm{L} - \beta x_0] = 0 \quad (3.14)$$
$$x'_0 = \gamma[x_0 - \beta(x(\mathrm{O2}) - \mathrm{L})] \quad (3.15)$$

where x_0 and x'_0 correspond to the times recorded by any clock in S and S', respectively, synchronised so that $x_0 = \gamma x'_0$. Defining $x1 \equiv x(\mathrm{O1})$, $x1' \equiv x'(\mathrm{O1})$, $x1_0 = x2_0 \equiv x_0$,

$x1'_0 = x2'_0 \equiv x'_0$, and introducing for O2 the coordinate transformations: $x2 \equiv x(\mathrm{O2}) - \mathrm{L}$, $x2' \equiv x'(\mathrm{O2}) - \mathrm{L}$, (3.12),(3.13) and (3.14),(3.15) are written, in a similar manner, as:

$$x1' = \gamma(x1 - \beta x1_0) = 0 \quad (3.16)$$
$$x1'_0 = \gamma(x1_0 - \beta x1) \quad (3.17)$$
$$x2' = \gamma(x2 - \beta x2_0) = 0 \quad (3.18)$$
$$x2'_0 = \gamma(x2_0 - \beta x2) \quad (3.19)$$

The world lines WL1 and WL2 referred to the coordinate systems, in the frame S, ($x1$,$x1_0$) and($x2$,$x2_0$), respectively, with origins at O_1 and O_2 are shown in Fig.3. Along the world lines, $x1 = \beta x1_0$ and $x2 = \beta x2_0$, the synchronisation condition $x1_0 = x2_0$ is relaxed so that $x1_0$ and $x2_0$ are allowed to vary independently in (3.16) and (3.18). Also shown are the loci of the positions of O1 and O2 (shown in the plot for $\beta = 1/3$) for other values of β. These are the hyperbolae:

$$x1_0^2 - x1^2 = (c\tau)^2 \qquad x1, x1_0 \geq 0 \quad (3.20)$$
$$x2_0^2 - x2^2 = (c\tau)^2 \qquad x2, x2_0 \geq 0 \quad (3.21)$$

The absence of any length contraction effect is made manifest by the constant separation, L, of the hyperbolae for any value of x_0 and β, including $\beta = 0$, when $x_0 = c\tau$.

Also clear, by inspection of Fig.3 is the absence of any relativity of simultaneity effect. Introducing the variables defined in Eqns(3.3)-(3.6) into Eqns(3.16)-(3.19) gives:

$$X1' = x1 \cos\theta - x1_0 \sin\theta \quad (3.22)$$
$$X1'_0 = x1_0 \cos\theta - x1 \sin\theta \quad (3.23)$$
$$X2' = x2 \cos\theta - x2_0 \sin\theta \quad (3.24)$$
$$X2'_0 = x2_0 \cos\theta - x2 \sin\theta \quad (3.25)$$

The $X1'$, $X1'_0$, $X2'$ and $X2'_0$ axes for $\beta = 1/3$ are drawn in Fig.3. Since:

$$\mathrm{O_1N_1} = X1'_0 = \mathrm{O_2N_2} = X2'_0 \quad (3.26)$$

when $x1_0 = x2_0$ it follows that, at that instant, $x1'_0 = x2'_0$ so that spatially separated events that are simultaneous in S (events on the world lines of O1 and O2 when $x1_0 = x2_0 = x_0$) are also simultaneous in S' — $x1'_0 = x2'_0 = x'_0$.

Indeed, simple inspection of Fig.3 shows that that the absence of any length contraction or relativity of simultaneity effect is a necessary consequence of translational invariance —the world line WL2 being obtained from WL1 by the spatial coordinate substitution $x \to x + \mathrm{L}$ (compare Eqns(3.12) and (3.13) with (3.14) and (3.15)).

In Fig.4 are shown world lines and coordinate axes for objects at rest in S' for different values of β, as well as the hyperbolic loci of the points shown on WL($\beta = 1/3$) and WL($\beta = 1/2$) for other values of β. At large values of x_0 the hyperbola approaches its asymptote, the projection into the x, x_0 plane of the light cone which is the world line WL($\beta = 1$). It is important to note that for a given orientation of transformed axes: X',X'_0; X'',X''_0:.. the value of β and hence the slope of the corresponding world line in

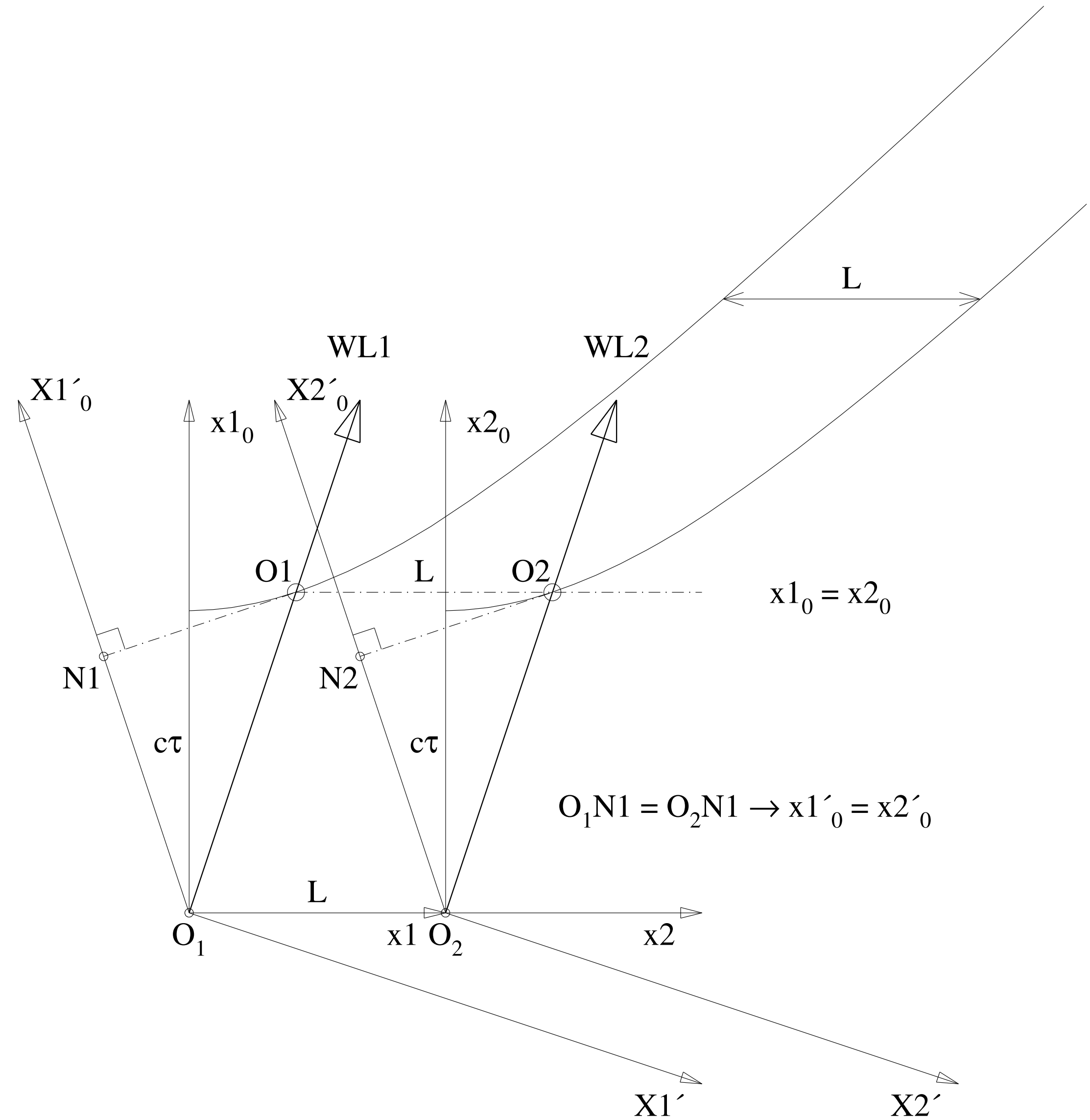

Figure 3: *Projection of simultaneous events in S on the world lines of O1 and O2 onto the $X1'_0$ and $X2'_0$ axes defined by Eqns(3.23) and (3.25). WL1 and WL2 correspond to $\beta = \tan\theta = 1/3$. World line points for O1 and O2 for other values of β, when $x1_0 = x2_0 = c\tau$ for $\beta = 0$, lie on the hyperbolic curves passing through the indicated positions of O1 and O2. The absence of any 'relativity of simultaneity' or 'length contraction' effects is evident from inspection of this figure, which shows invariance under the transformation of spatial coordinates $x \rightarrow x + \mathrm{L}$, for O1 $\rightarrow$ O2.*

S is fixed. The X',X'_0 axes correspond to $\beta = 1/3$ and the X'',X''_0 axes to $\beta = 1/2$. It is nonsensical to draw on the plots world lines with any other slope. As will be seen in the following section, just this fundamental error was made by Minkowski in his original 'space-time' paper [5], and has since been universally followed by authors of text books and the pedagogical literature.

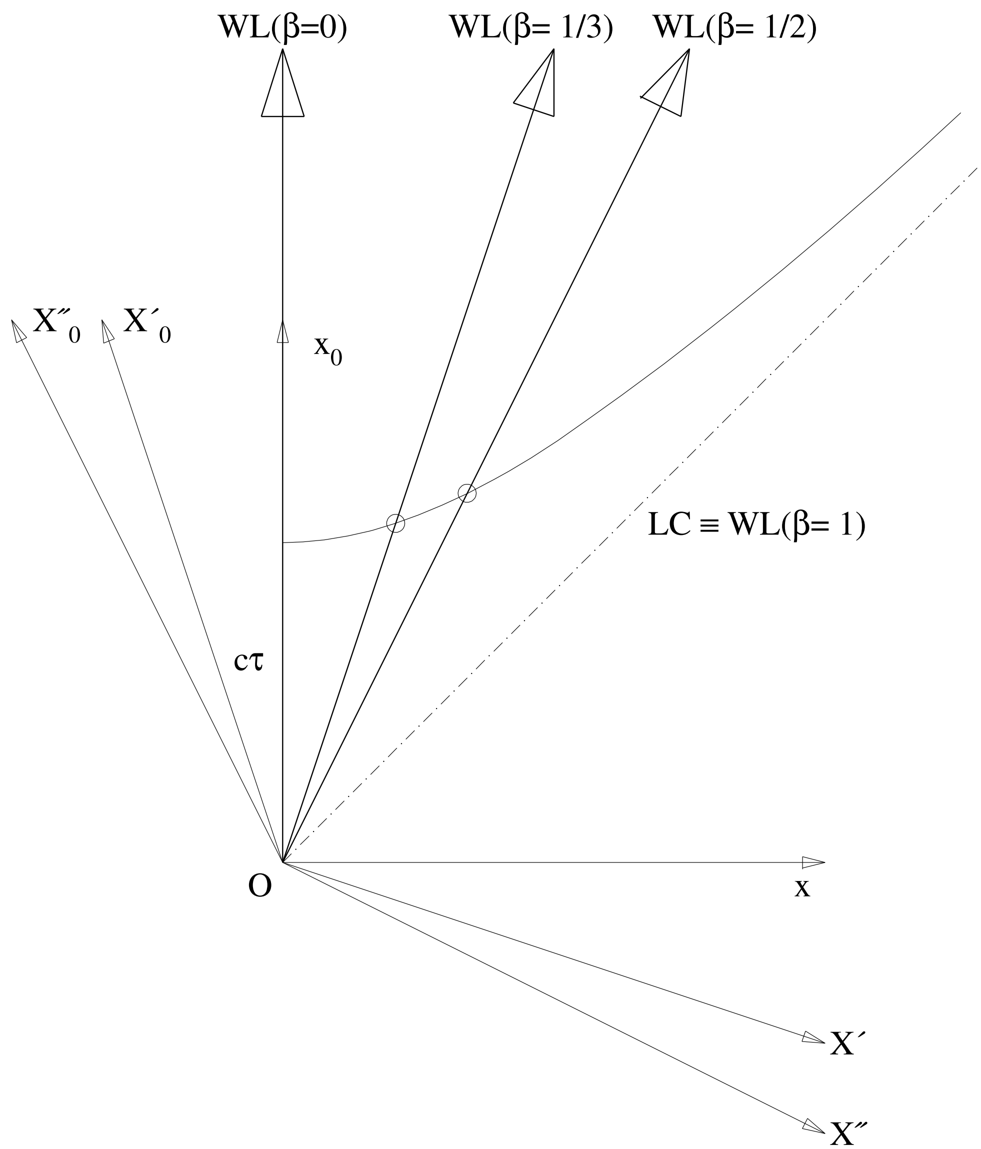

Figure 4: *Coordinate axes in S' for different world lines in S. There is a one-to-one correspondence between world line directions and the orientation of the axes in S': X',X'_0 correspond to $\beta = 1/3$ and X'',X''_0 to $\beta = 1/2$. Drawing any other world lines for the S' coordinate axes shown is a nonsensical procedure.*

4 Minkowski's spurious 'length contraction' effect

A fair copy of Fig.1 of Ref. [5], used by Minkowski to derive a 'length contraction' effect, is drawn in Fig.5. Only Minkowski's time coordinates t,t' are replaced by x_0,x'_0 and, for clarity, the intersection of the line A'B' —the tangent to the hyperbola $\mathrm{AO}^2 = x_0^2 - x^2$ at A'— with the x_0 axis, has been labelled E'. Minkowski assumes that A', a point on the world line of the orgin of S', lies on the x'_0 axis, for non-zero values of x'_0 . This is wrong. If the world line of the origin of S', as viewed in S, is $x = \beta x_0$. as assumed by Minkowski, the x', x'_0 axes must be drawn parallel to the X', X'_0 axes shown in Fig.4 above. The correct world line for the x', x'_0 axes of Fig.5a is $x = -\beta x_0$, as shown in Fig.6. In fact the hyperbola giving the locus of the coordinates of the origin of S', for different values of β, can never cross the x'_0 axis, as shown in Fig.5a. The LT is not mentioned in Minkowski's paper —only geometric properties of the hyperbolae which specify time-like or space-like invariant intervals. To calculate coordinates from invariant interval relations square roots must be taken, leading to sign ambiguities. Insufficient attention to the correct sign choice in correlating world lines with axis directions is at the origin of Minkowski's error.

In order to calculate a 'length contraction' effect Minkowski assumed an oblique projection as shown in Fig.5b. Thus the length Q'Q' of an object lying along the x' axis corresponds to the length QQ along the x-axis, the lines QQ' being drawn parallel to the x'_0 axis. Such a projection is seen to result in a length contraction effect of QQ as compared to Q'Q'. However, the geometry of Fig.5b gives:

$$\mathrm{QQ} = \frac{\mathrm{Q'Q'}(1-\beta^2)}{\sqrt{1+\beta^2}} \tag{4.1}$$

not the conventional length contraction effect which would instead give: $\mathrm{QQ} = \mathrm{Q'Q'}\sqrt{1-\beta^2}$. In order to arrive at the latter result, Minkowski makes the following hypotheses connecting the geometries of Fig.5a and Fig.5b:

$$\mathrm{PP} = \ell \mathrm{OC} \tag{4.2}$$
$$\mathrm{Q'Q'} = \ell \mathrm{OC'} \tag{4.3}$$
$$\mathrm{QQ} = \ell \mathrm{OD'} \tag{4.4}$$

where ℓ is the length along the x' axis of an object at rest in S' and observed in S', and uses a different geometrical definition of the length contraction effect. Minkowski's arguments are based on the geometry of Fig.5b. However the axis Ox' in the figure is inconsistent with the world lines PP' of the ends of an object at rest in S. For such an object $\beta = 0$ and the x and x' axes coincide. Taking the ratio of (4.4) to (4.2) gives:

$$\frac{\mathrm{QQ}}{\mathrm{PP}} = \frac{\mathrm{OD'}}{\mathrm{OC}} \equiv f_{LC} \tag{4.5}$$

Minkowski now assumes that f_{LC} is the length contraction factor. Since the line E'A'B' is drawn as a tangent to the hyperbola, it follows that the tangent of the angle AE'A' is $1/\beta$ and the line segment E'A'B' is parallel to the x' axis. From the symmetry of the figure about the light cone projection OBB', the angles AE'A' and CD'C' are equal. It then follows from the geometry of Fig.5a that $\mathrm{OD'} = \mathrm{OC}\sqrt{1-\beta^2}$ giving, with (4.5), the relation

$$\mathrm{QQ} = \mathrm{PP}\sqrt{1-\beta^2} \tag{4.6}$$

or $f_{LC} = \sqrt{1-\beta^2}$.

The argument given attempts to relate the world lines of the ends an object at rest a)

b)

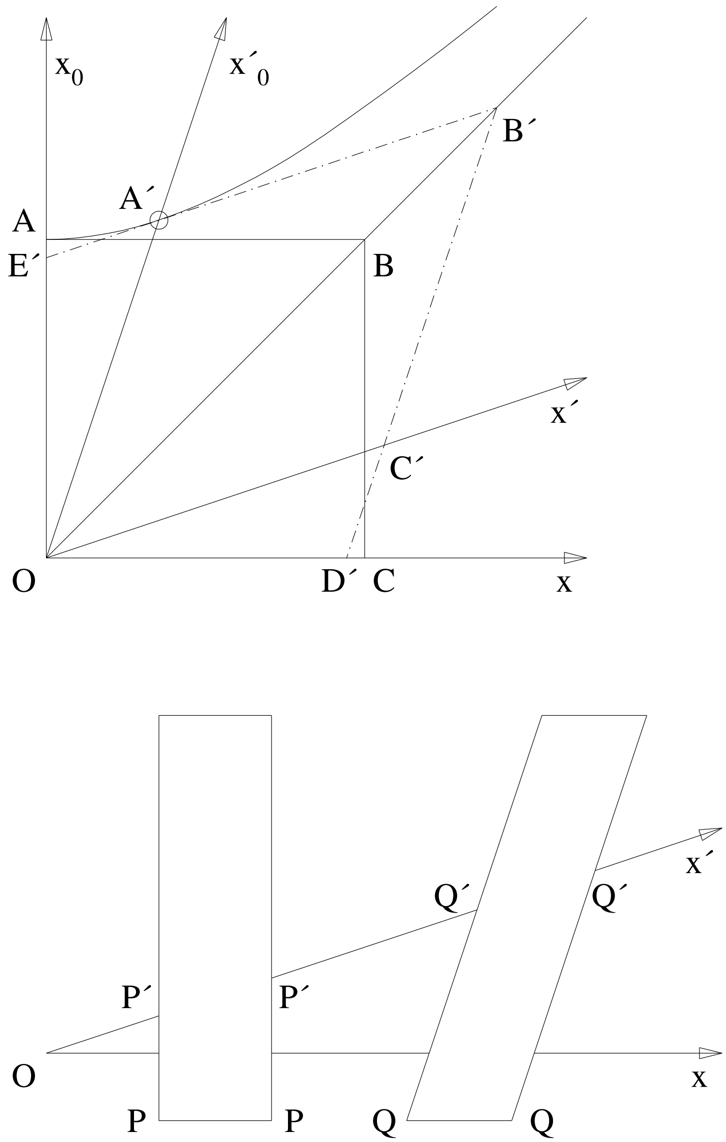

Figure 5: *Copy of Minkowski's figure from Ref. [5]. See text for discussion.*

in S, represented by PP', to the world lines, QQ', of the ends of an object at rest in S', on the assumption that the x' axis is as drawn in Fig.5a. This is a incorrect procedure since the axes needed to specify the world line, in the frame S', of an object at rest in the frame S, are not as shown in Fig.5b, but as in Fig.6 with exchange of primed and unprimed coordinates. This space-time plot is discussed further in the following section (see Fig.13c below). The fundamental error in Fig.5a of incorrectly plotting the world line of the origin of S' along the x'_0 axis, instead of as in Fig.6, has been systematically repeated in almost all text book discussions of the Minkowski plot. It will be seen however that the argument typically used to obtain the length contraction effect, although, finally, geometrically identical to Minkowski's calculation, does not invoke the world lines PP' of an object at rest in S, but considers, instead of the time-like interval relation invoked by Minkowski (the hyperbola through A and A'), a space-like one involving directly the length of the considered object.

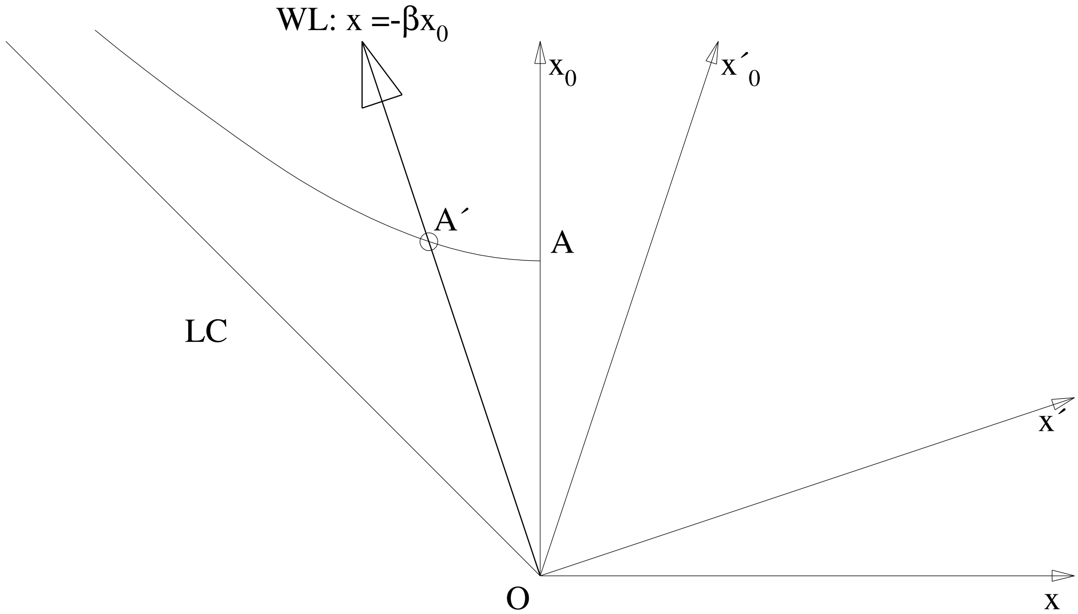

Figure 6: *Correct world line in S of the origin of S' for Minkowski's choice of x and x'_0 axes as in Fig.5a. Compare with the incorrect world line OA' in the latter figure.*

5 Text book treatments of the Minkowski plot

For definiteness, the discussion of the Minkowski plot in the widely-known book 'Space-time Physics' by Taylor and Wheeler [7] will be first considered. Similar treatments are to be found in the books of Aharoni [8], Panofsky and Philips [9] and Mermin [10]. Minkowski's sign error in drawing the directions of the x' and t' axes has been replicated in many text books treating special relativity. A survey of such books in the library of the 'Section de Physique' at the University of Geneva and the CERN library, turned up 24 books with the error, including notably the first volume of 'The Feynman Lectures in Physics' [11]. The only exception found was a book by Anderson [12] where the world line of S' corresponding to the x', x'_0 axes as drawn in Fig.5 is correctly given as $x = -\beta x_0$

corresponding to the LT: $x' = \gamma(x + \beta x_0) = 0$. However, this author does not use the Minkowski plot to discuss time dilatation or 'length contraction', but rather the analysis of light signals observed in different inertial frames.

Fig.7, which is a combination of Figs.66 and 69 of Ref. [7] is the basis for discussion of both time dilatation and 'length contraction' in this book. An object at rest in S', lying along the x' axis, is considered and the world lines of the ends $\sharp$ 1 and $\sharp$ 2 of the object are drawn, end $\sharp$ 1 being at the origin of S'. As in the original Minkowski plot, Fig.5a, the world line of the origin in S' is incorrectly assumed to lie along the x'_0 axis instead of on the other side of the x_0 axis, as in Fig.6.

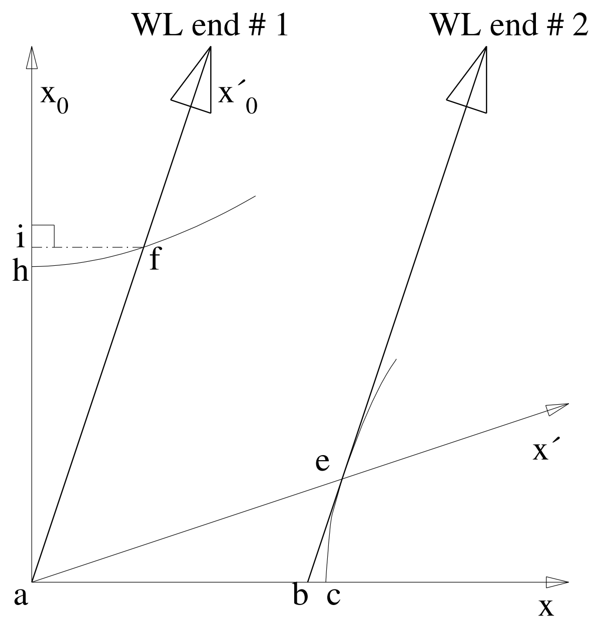

Figure 7: *Figures used in Ref. [7] to discuss the time dilatation and 'length contraction' effects. See text for discussion.*

In order to calculate the time dilatation effect it is assumed that the distance af in the frame S represents one unit of time in the frame S', and that the distance ah represents one unit of time in the frame S. Since h and f lie on the hyperbola:

$$(ah)^2 = (\Delta x_0)^2 - (\Delta x)^2 = (ai)^2 - (if)^2 = (ai)^2(1 - \tan^2\theta) = (ai)^2(1 - \beta^2) \qquad (5.1)$$

it follows that

$$\frac{af}{ah} = \frac{ai}{ah\cos\theta} = \sqrt{\frac{1+\beta^2}{1-\beta^2}} \equiv F \qquad (5.2)$$

so that one unit of time in the frame S' is assumed to correspond to F units of time along the x'_0 axis in the frame S [13]. To derive the time dilatation effect it is further assumed that the time in S of the event at f in S' is given by the orthogonal projection fi onto the x_0 axis. The time dilatation factor is then found to be, using (5.2) and (3.3):

$$\frac{\Delta x_0}{\Delta x'_0} = \frac{ai}{ah} = \frac{af\cos\theta}{ah} = \frac{1}{\sqrt{1-\beta^2}} = \gamma \qquad (5.3)$$

For direct comparison with the calculation of Section 2 above, using the correct projection procedure in the x, x_0 plane, it is convenient to introduce explicitly into the plot the scale factor of Eqn(5.2) relating a coordinate $\tilde{X}'_0$ to x'_0:

$$\tilde{X}'_0 = \sqrt{\frac{1+\beta^2}{1-\beta^2}} x'_0 \equiv F x'_0 \tag{5.4}$$

The scale factor relating $\tilde{X}'_0$ to x'_0 is the reciprocal of that relating X'_0 to x'_0 in Eqn(3.6). The calculation of the time dilatation relation (5.3), using the orthogonal projection onto the x_0 axis, is shown in Fig. 8a, where WL1 denotes the world line of end $\sharp$ 1 of the object. The geometry of this figure shows that:

$$\Delta x'_0 = \frac{\Delta \tilde{X}'_0}{F} = \frac{\Delta x_0}{F\cos\theta} = \frac{\Delta x_0}{\gamma} \tag{5.5}$$

in agreement with (5.3). The correct calculation of the time dilatation effect with the world line of the end $\sharp$ 1 of the object on the opposite side of the x_0 axis, as in Fig.6, is shown in Fig. 8b. The orthogonal projection is now from the event onto the X'_0 axis which is scaled up by the factor F to obtain the x'_0 coordinate according to Eqn(3.6). Since the geometry of Fig.8b is related to that of Fig.2 by reflection in the x_0 axis, the same result, (3.11), identical to (5.5) is obtained. By use of a different projection (orthogonal from the event onto the x_0 axis, instead of onto the X'_0 axis) and a different scaling factor for time intervals between S and S' (the reciprocal of that, (3.6), obtained from the LT) the correct time dilatation formula is obtained in Fig.8a, from the incorrectly drawn world lines of Figs.7 and 8a. As will now be seen, this fortuitous cancellation of three different errors is no longer applicable in the discussion of the transformation of spatial intervals.

By considering the hyperbola passing through the points e and c in Fig.7, a relation analogous to 5.2 is derived:

$$\frac{ae}{ac} = \sqrt{\frac{1+\beta^2}{1-\beta^2}} \tag{5.6}$$

If $\Delta x'$ is the distance between a and e in the frame S', it is now assumed, as for the discussion of the time coordinates in the time dilatation effect, that

$$ae = \sqrt{\frac{1+\beta^2}{1-\beta^2}} \Delta x' \tag{5.7}$$

In order to obtain the corresponding length interval in the frame S, an oblique projection, parallel to the x'_0 axis is taken, instead of the orthogonal one, perpendicular to the x_0-axis, assumed for the transformation of time intervals between S' and S. This gives:

$$\Delta x = ab \tag{5.8}$$

Combining (5.7) and (5.8) gives:

$$\Delta x = \frac{ab}{ae}\sqrt{\frac{1+\beta^2}{1-\beta^2}} \Delta x' \tag{5.9}$$

Note that this definition of the 'length contraction' effect is not the same as that assumed by Minkowski. In terms of the geometrical definitions of Fig.7, Minkowski assumes that:

$$\Delta x = f_{LC}\Delta x' = \frac{ab}{ac}\Delta x' \tag{5.10}$$

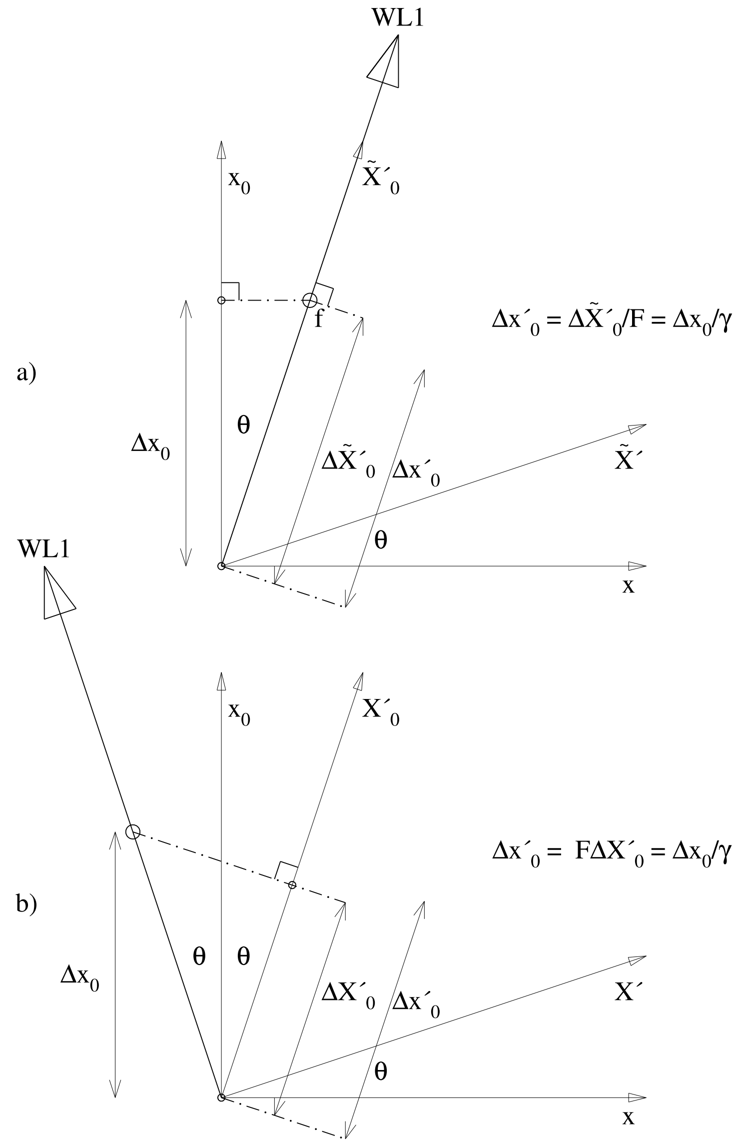

Figure 8: *a) Calculation of time dilatation according to Ref. [7]. b) Correct calculation of time dilatation from the LT as in Fig.2 above. See text for discussion.*

Therefore no scale factor connecting lengths observed in the frame S' to those observed in the frame S is introduced by Minkowski.

It is found from the geometry of Fig.7 that

$$\frac{ab}{ae} = \frac{1-\beta^2}{\sqrt{1+\beta^2}} \tag{5.11}$$

so that

$$\Delta x = \sqrt{1-\beta^2}\Delta x' = \frac{\Delta x'}{\gamma} \tag{5.12}$$

This is the 'length contraction' effect calculated according to Ref. [7]. A comparison with a calculation where the world lines of the ends of an object consistent with the x', t' axes drawn in Fig.7 and using the correct projection procedure obtained from the LT is shown in Fig.9. For ease of comparison, the variable $\tilde{X}'$ analogous to $\tilde{X}'_0$ in Eqn(5.4) is introduced:

$$\tilde{X}' = \sqrt{\frac{1+\beta^2}{1-\beta^2}}x' = Fx' \tag{5.13}$$

The derivation of Eqn(5.12), as just described, is shown in Fig. 9a

In Fig. 9b the world lines WL1, WL2 of end ♯ 1 and end ♯ 2 of the object are shown in the frame S for $\beta = 0$ and $\beta = 1/3$. Inspection of Fig. 9b, where the correct projections as derived from the LT, which in the present case are (c.f. Eqns(3.32)-(3.25)):

$$X1' = x1\cos\theta + x1_0\sin\theta \tag{5.14}$$
$$X1'_0 = x1_0\cos\theta + x1\sin\theta \tag{5.15}$$
$$X2' = x2\cos\theta + x2_0\sin\theta \tag{5.16}$$
$$X2'_0 = x2_0\cos\theta + x2\sin\theta \tag{5.17}$$

are used, shows that, as in the analogous Fig. 3, the length of the object in S' (L') is the same as in S (L) and that there is no 'relativity of simultaneity' effect. As previously shown in Section 2 above, the length of an object (by definition a $\Delta x_0 = \Delta x'_0 = 0$ projection) is a Lorentz invariant quantity — there is no 'length contraction'. For the transformation of spatial intervals there is evidently no fortuitous cancellation of the errors of drawing wrongly the world lines of the ends of the object, using an oblique projection, and a scaling factor that differ from the relations (5.14),(5.15) and (3.5),(3.6) obtained directly from the LT.

Inspection of Fig.7 shows that Ref.[7] (in common with other text books) uses different projections in the discussions of time dilatation and 'length contraction' —orthogonal to the x_0 axis for time dilatation and oblique, parallel to the x'_0 axis, for 'length contraction'. Given the symmetry of the LT equations with respect to exchange of x,x_0 and x',x'_0 also manifest in the correct projection equations (3.7) and (3.8) or (5.14) and (5.15), there can be no physical justification for the arbitary choice of projections that is made. Using, in each case the other type of projection would give time contraction and length dilatation effects, both in conflict with the correct predictions of the LT —time dilatation but no 'length contraction'— as well as, in the former case, contradiction to experiment.

This point is illustrated in Fig.10 where the transformation of time is calculated using: in a) a projection normal to the x_0 axis as in Ref. [7], in b) an oblique projection as used

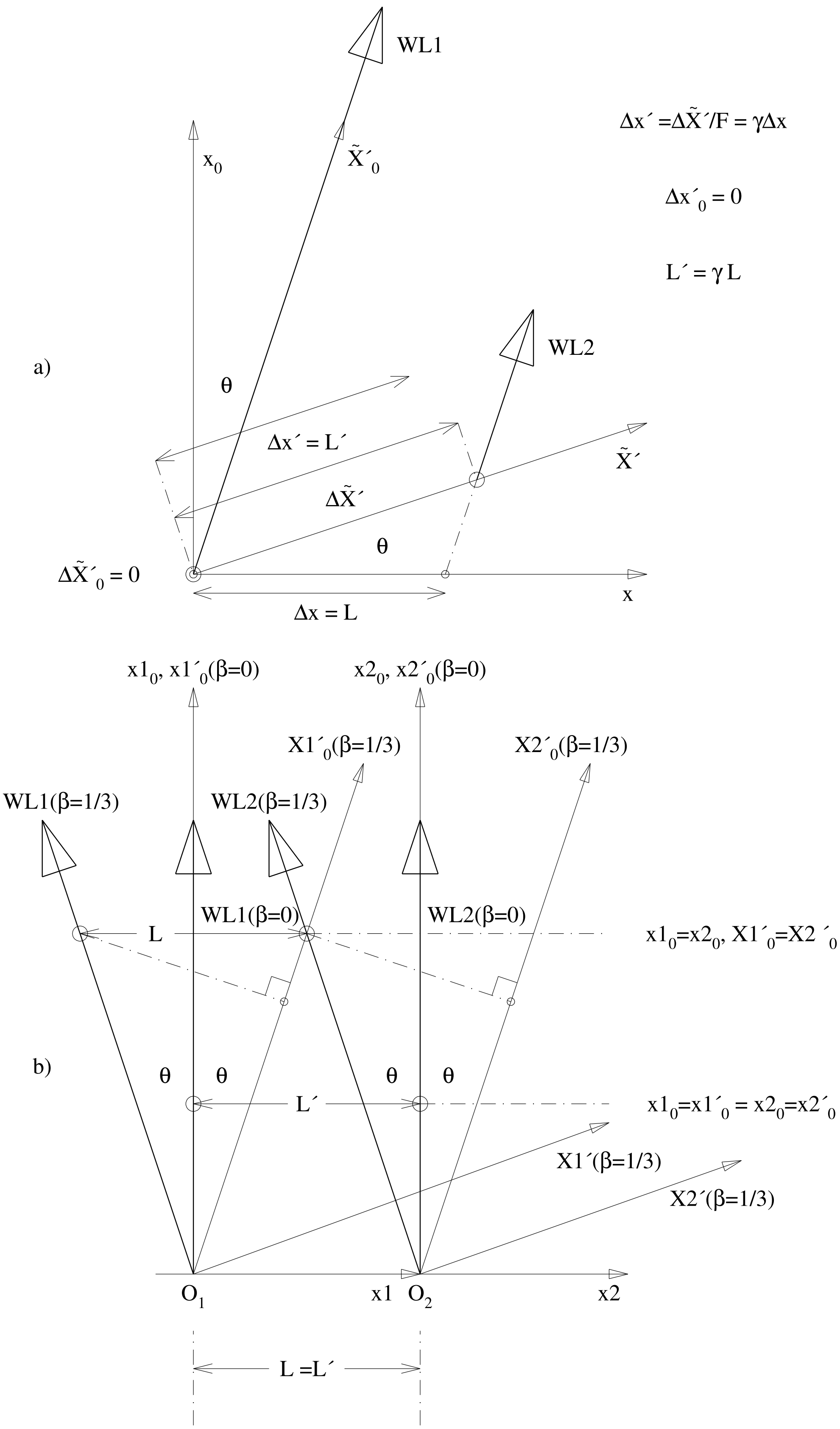

Figure 9: *a) Calculation of 'length contraction' according to Ref. [7]. b) Correct prediction of the LT showing the invariance of the length of the object and the absence of any 'relativity of simultanenity' effect. See text for discussion.*

in Ref. [7] to derive 'length contraction' and in c) the correct projections of the LT using Eqns(3.5)-(3.8), *mutatis mutandis* to account for the different direction of the world line. Fig.10b corresponds to 'time contraction'. Application of the perpendicular projection of Fig.10a to the x-axis would give a 'length dilatation' effect. In the correct relativistic analysis of the transformation properties of time and length intervals shown in Fig.8b and Fig.9b, respectively, the same projection and scaling operations are used in both calculations.

The discussion of 'relativity of simultaneity' based on Fig.67 of Ref. [7] is illustrated in Fig.11. The events 1 and 2 have the same x'_0 coordinates, but $x2_0 > x1_0$. The events 3 and 4 have the same x_0 coordinates but $x'3_0 > x4'_0$. A similar argument applied to Fig.9b where the correct projective geometry of the LT is applied would seem to indicate that simultaneous events in the frame S at the ends $\sharp$ 1 and $\sharp$ 2 of the object considered are not simultaneous in S', since the projections onto the $X1'_0$ axis give $X1'_0(\mathrm{end}\sharp 2) > X1'_0(\mathrm{end}\sharp 1)$. Such a 'relativity of simultaneity' effect would, however, be in direct contradiction to translational invariance of the time dilatation relation: $\Delta t = \gamma \Delta t'$, pointed out in Section 2 above. To understand this apparent contradiction it is necessary to examine more closely the operational meaning of 'relativity of simultanenty' in an actual experiment. Clearly, at least two spatially separated clocks must be introduced into any such discussion, whereas, in the examples just discussed, based on Fig.11 or Fig.9, only a single clock — the one at the end $\sharp$ 1 of the object— is considered. Introducing clocks at both ends of the object, synchronised in S,(i.e. clocks recording the same time, $t = x_0/c$, at any instant) and denoting the corresponding times in the frame S' by $t'_1 = x1'_0/c$ and $t'_2 = x2'_0/c$, the oblique projection of Fig.11 gives the space-time configuration shown in Fig.12a, where different scaled time and space coordinates, as in Eqn(5.4) and (5.13), have been introduced for end $\sharp$ 1 and end $\sharp$ 2 of the object. It is clear from the geometry of this figure that, due to TD, for any pair of events on the world lines of end $\sharp$ 1 and end $\sharp$ 2 in S for which $\Delta x_0 = 0$, then $\tilde{X}1'_0 = \tilde{X}2'_0$ and hence $x1'_0 = x2'_0$ —events which are simultaneous in S are also simultaneous in S'— there is no 'relativity of simultaneity' in this case. The incorrect procedure (according to the LT) of obliquely projecting an event on the world line of end $\sharp$ 2 onto the $\tilde{X}1'_0$ axis, as in Fig.12a, gives the interval $\Delta\tilde{X}'_0$, but is not indicative of any 'relativity of simultaneity' of clocks, at the ends of the object, synchronised in the frame S, when observed from the frame S' In fact, in Fig.12a, oblique projections of events 1 and 2 onto the $\tilde{X}1'_0$ or $\tilde{X}2'_0$ axes give the four time intervals $\mathrm{O_1M}$, $\mathrm{O_1}1$, $\mathrm{O_2}2$ and $\mathrm{O_2L}$, but only $\mathrm{O_1}1 = \tilde{X}1'_0 = \mathrm{O_2}2 = \tilde{X}2'_0$ represent the times of the clocks in S' The spurious 'relativity of simultaneity' effect of Fig.11 arises from falsely identifying either $\mathrm{O_1M}$ or $\mathrm{O_2L}$ with corresponding clock times in S'.

Similar conclusions are drawn from Fig.12b, analogous to Fig.5, where the correct projection procedure relating coordinates in the frames S and S' is used. Orthogonal projections of the points 1 and 2 on the world lines of end $\sharp$ 1 and end $\sharp$ 2 of the object onto either the $X1'_0$ or $X2'_0$ axes define the four time intervals $\mathrm{O_1P}$, $\mathrm{O_1Q}$, $\mathrm{O_2N}$ and $\mathrm{O_2R}$ which are related by the condition

$$\Delta X'_0 = \mathrm{O_1Q} - \mathrm{O_1P} = \mathrm{O_2R} - \mathrm{O_2N} \tag{5.18}$$

However, only two of these intervals, $\mathrm{O_1P} = X1'_0$ and $\mathrm{O_2R} = X2'_0$ correspond to the times registered by the clocks in S' at time x_0 in the frame S. The spurious 'relativity of simultaneity' effect arises from incorrectly identifying either of the time intervals $\mathrm{O_1Q}$ or

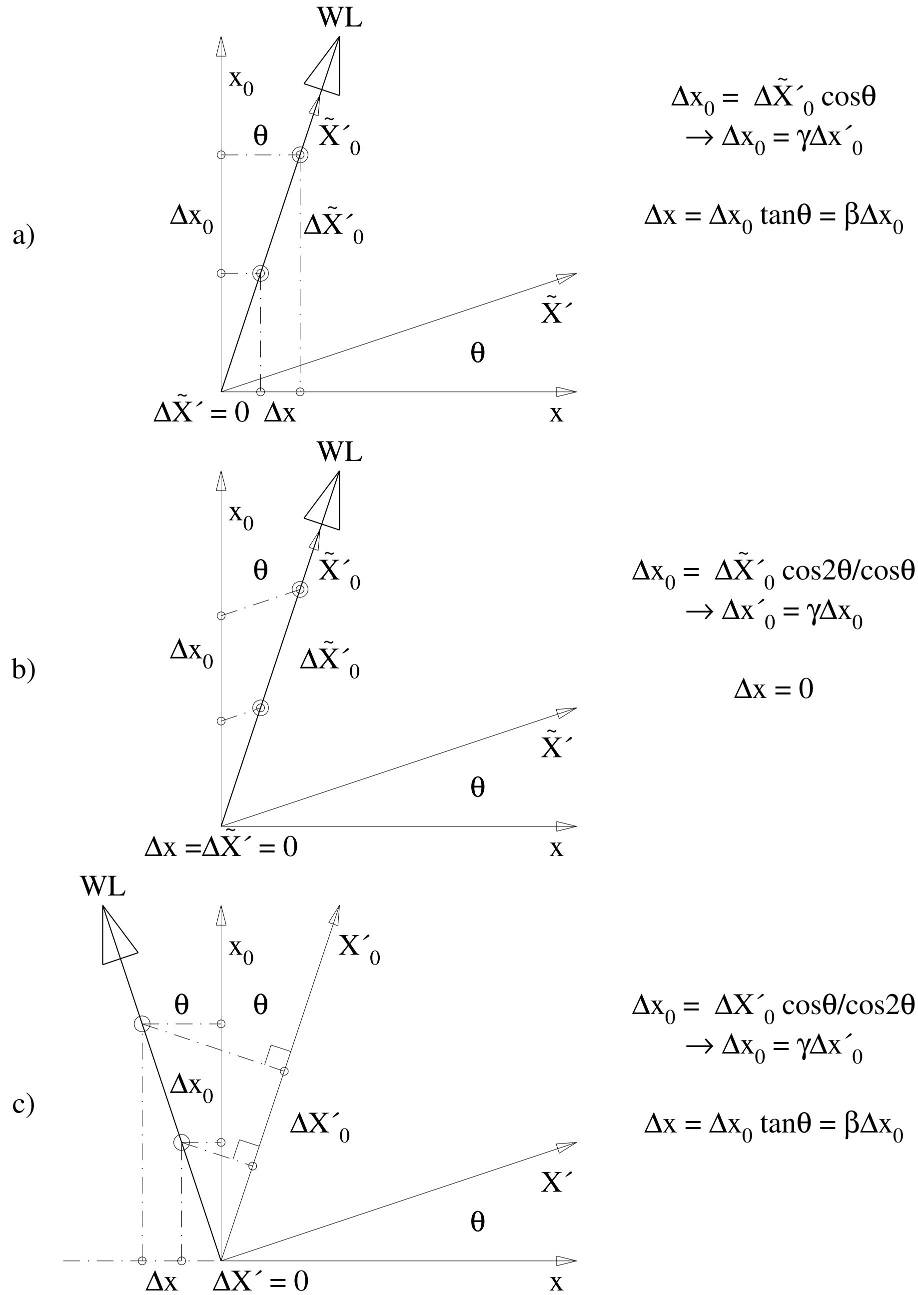

Figure 10: *Time transformation using different projection procedures: a) perpendicular to the x_0 axis as in Ref. [7], b) oblique, parallel to the $\tilde{X}'$ axis, c) Correct prediction of the LT. Fortuitously, a) and c) give the same prediction.*

O_2N with the observed time in S' of an event at time x_0 in S.

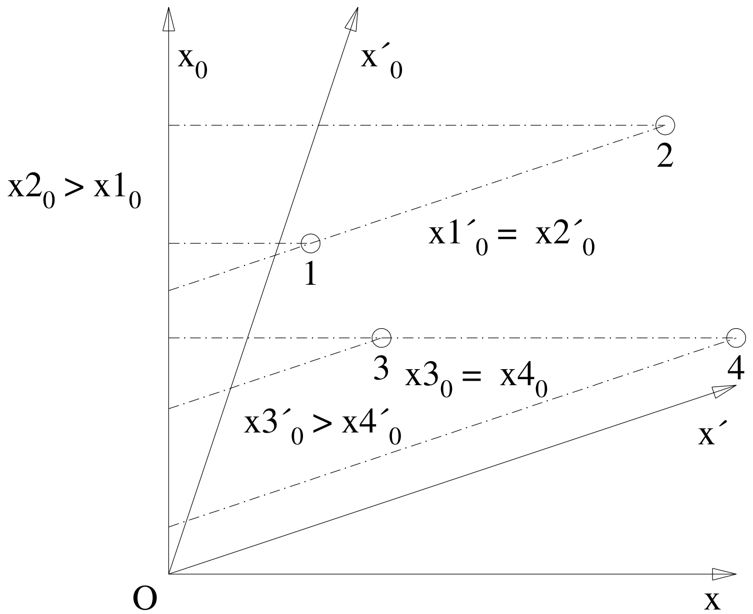

Figure 11: *Figure showing apparent 'relativity of simultaneity' from Ref. [7]. See text for discussion.*

The non-existent 'relativity of simultaneity' effect then results, unlike 'length contraction', not from geometrical errors in the Minkowski plot but from a fundamental misunderstanding of the physical meaning of time intervals corresponding to certain geometrical projections, regardless of whether the latter are correct (as predicted by the LT) as in Fig.12b, or not, as in Fig.12a. It is essential that two spatially separated and synchronised clocks, each with its own time reading, be introduced into the problem in order to perform any meaningful analysis of it. This is not the case for Fig.11, or the equivalent Fig.67 of Ref. [7].

The discussion of 'length contraction' in Mermin's book 'Space and Time in Special Relativity' [10] is similar to that of Taylor and Wheeler [7]. Oblique projections are used and the t' axis is incorrectly drawn, following Minkowski, along the world line in S of the origin of the frame S'. The geometry of Fig.17.24 of Ref. [10], illustrating Mermin's interpretation of 'length contraction' is the same as that of Fig.9a of the present paper and the same incorrect result (5.12) is obtained.

Mermin's discussion of 'relativity of simultaneity' is, however, somewhat different to that of other authors. Instead of considering, as in Ref. [7] the abstract geometrical projections of Fig.11, the world lines of two clocks at rest in the frame S are considered, which at least, unlike Fig.11, satisifies the minimum condition for a meaningful discussion of the problem. In Fig.13a is shown a fair copy of Fig.17.22 of Ref. [10], the basis of the discussion there of 'relativity of simultaneity'. The S frame coordinates $\tilde{X}'$ and $\tilde{X}'_0$ related to x' and x'_0 by the relations (5.13) and (5.4), repectively, also given by Mermin, are shown. The clocks are synchronised at $x_0 = 0$ and it is shown that at the later time when the world line of clock 2 crosses the $\tilde{X}'$ axis an observer in S' will see that it is in advance of the clock 1 by the time interval, Δx_0 that has elapsed in the frame S. This is supposed to be evident from the figure (perhaps from an oblique projection parallel

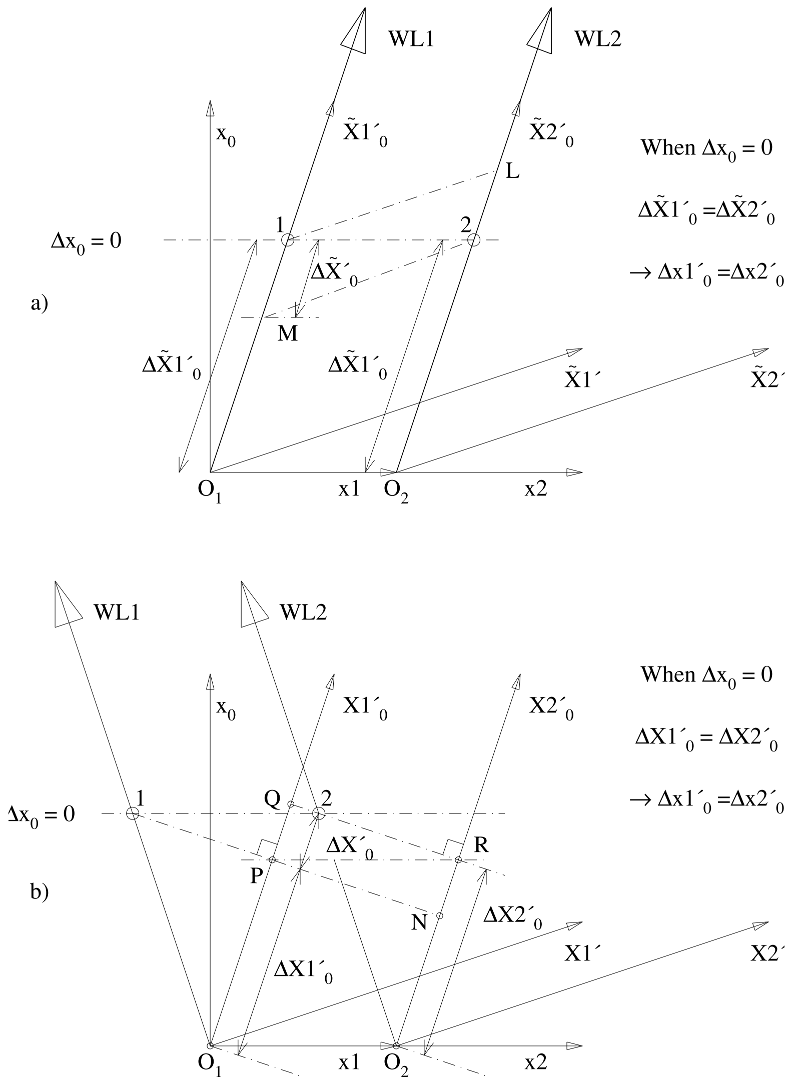

Figure 12: *Discussion of simultaneity introducing the world lines of separate clocks at the ends of the object discussed in Ref. [7]: a) oblique projections similar to that performed on points 3 and 4 in Fig.11, b) correct prediction of the LT. In neither case is there any 'relativity of simultaneity' effect. See text for discussion.*

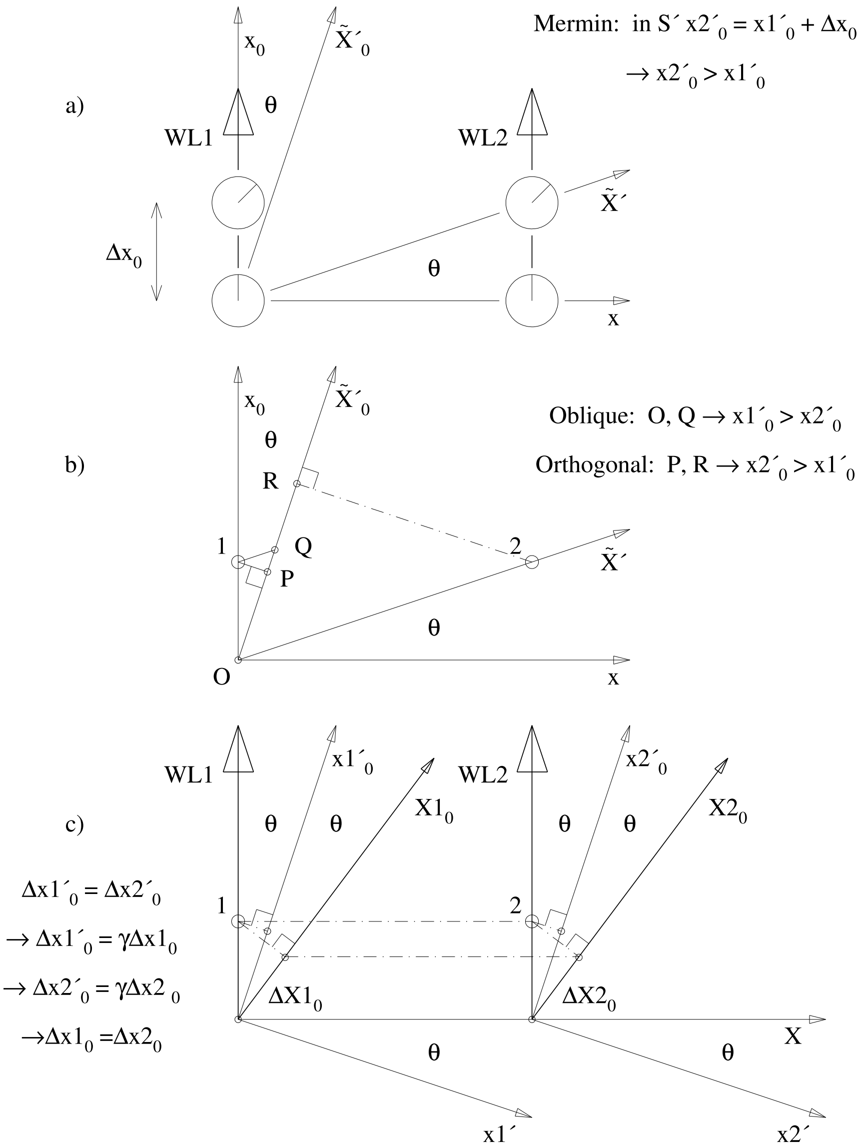

Figure 13: *a) Figure claiming to demonstrate 'relativity of simultaneity' from Ref. [10], b) Naive 'relativity of simultaneity' effects given by oblique or perpendicular projections as in Fig.11. c) correct calculation of the time transformation corresponding to clocks at rest in S, as shown in a). See text for discusion.*

to the $\tilde{X}'$ axis?) since no supporting explanation or calculation is given. The claimed 'relativity of simultaneity' effect is such that $x2'_0 > x1'_0$. Comparing in Fig.13b different projections of simultaneous events in S on the world lines of the clocks, shows that an oblique projection, as used in Mermin's discussion of 'length contraction', gives instead $x1'_0 > x2'_0$. Projection orthogonal to the x_0 axis onto the $\tilde{X}'_0$ axis gives simultaneous events in S'. Indeed, Fig.13b indicates that in order to obtain $x2'_0 > x1'_0$ a projection orthogonal to the $\tilde{X}'_0$ axis is required. Such a projection is nowhere else mentioned in Ref. [10].

The correct simultaneity analysis, according to the LT, of events on the world lines of clocks at rest in S is shown in Fig.13c, which is analogous to Figs.5 and 12b. Since the clocks are at rest in the frame S, their world lines in the frame S' are: $x1' = -\beta x1'_0$ and $x2' = -\beta x2'_0$. The Minkowski plot given by the LT in the frame S' is then the same as Fig.6, but with exchange of primed and unprimed coordinates. As can be seen in Fig.13c, The x' and x'_0 axes are orthogonal as a conseqence of the relations, the inverse of (2.7) and (2.8):

$$X = x' \cos\theta + x'_0 \sin\theta \quad (5.19)$$
$$X_0 = x'_0 \cos\theta + x' \sin\theta \quad (5.20)$$

while the X and X_0 axes are obliquely oriented. As evident, in any case, from translational invariance, the projections in Fig.13c show that there is no 'relativity of simultaneity' effect in the problem. Notice that, in this case, the TD effect is inverted with respect to that shown in Fig.2 above; i.e. the clocks in the frame S' appear, to an observer in this frame, to be running faster than those in S. The space-time experiment shown in Fig.13c is thus the reciprocal of that in Fig.2.

6 Summary

The manifest translational invariance of the TD relations in (2.13) demonstrate the spurious nature of correlated 'relativity of simultaneity' and 'length contraction' effects that have hitherto, following Einstein [1], been derived from the LT, as explained in previous papers by the present author [2, 3, 4] and Section 2 of the present paper.

Alternative derivations of 'length contraction' from the geometry of the Minkowski plot are shown to be flawed by a sign error in an angle, when drawing the x' and t' axes on the original Minkowski plot [5], which has been propagated, uncorrected, in essentially all text-book treatments of the subject, as well as the use projection operations at variance with those required by the LT. When the correct projection and scaling operations derived in Section 3, directly from the LT, are applied, it is clear, by simple inspection, (see Fig. 3 and Fig. 9b) that there is no 'length contraction' effect.

The 'relativity of simultaneity' effect derived from the Minkowski plot, unlike that obtained in Eqns(2.14)-(2.17), directly from misuse of the LT, is not a direct consequence of the 'length contraction' effect obtained from the same plot. It is, instead, produced by a false identification of certain projections on the plot with the times recorded by two spatially separated and synchronised clocks. It is shown in Fig.12a, that, even with an incorrect oblique projection procedure, similar to that used to obtain 'length contraction', there is no corresponding 'relativity of simultaneity' effect.

The present author has been able to find only one text book in which the x' and t' axes of the Minkowski plot are correctly drawn [12]. However, the plot is not used in the book in the discussion of the 'length contraction' effect.

References

[1] A.Einstein, Annalen der Physik **17**, 891 (1905).
English translation by W.Perrett and G.B.Jeffery in ‘The Principle of Relativity’ (Dover, New York, 1952) P37, or in ‘Einstein’s Miraculous Year’ (Princeton University Press, Princeton, New Jersey, 1998) P123.

[2] J.H.Field, ‘Spatially-separated synchronised clocks in the same inertial frame: Time dilatation, but no relativity of simultaneity or length contraction’ arXiv pre-print: http://xxx.lanl.gov/abs/0802.3298.

[3] J.H.Field, ’Clock rates, clock settings and the physics of the space-time Lorentz transformation’, arXiv pre-print: http://xxx.lanl.gov/abs/physics/0606101.

[4] J.H.Field, ’Translational invariance and the space-time Lorentz transformation with arbitary spatial coordinates’, http://xxx.lanl.gov/abs/physics/0703185.

[5] H.Minkowski, English translation of Address to the 50th Association of German Natural Scientists and Physicians, Cologne, 21st September 1908, in ‘The Principle of Relativity’ (Dover, New York, 1952) P75.

[6] See P46 in the second reference in [1], and P134 in the third one.

[7] E.F.Taylor and J.A.Wheeler, ‘Spacetime Physics’, W.H.Freeman and Company, San Franciso 1966, Section 48, P92.

[8] J.Aharoni, ‘The Special Theory of Relativity’, (O.U.P., Oxford, 1965), Ch9, P30.

[9] W.H.Panofsky and M.Phillips, ‘Classical Electricity and Magnetism’, 2nd Edition (Addison-Wesley, Cambridge Mass, 1962) Section 16-4, P397.

[10] N.D.Mermin, ‘Space and Time in Special Relativity’ (McGraw Hill, New York, 1968) Ch17, P155.

[11] R.P.Feynman, R.B.Leighton and M.Sands, ‘The Feynman Lectures in Physics’ (Addison-Wesley, Reading Massachusetts, 1963) Vol I Ch17, P17-1, Fig.17-2.

[12] J.L.Anderson, ‘Principles of Relativity Physics’ (Academic Press, New York, 1967) Section 6.6, P151.

[13] Ref. [10], P174.